Ex 2.1

Q1)

$$z = \frac{4+3i}{3-4i} = \frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$$

$$= \frac{12+16i+9i+12i^2}{9-16i^2}$$

$$= \frac{25i}{25}$$

$$= i$$

$\therefore Re(z) = 0$, $Im(z) = 1$

Q2) $z = e^{i\theta} - e^{-i\theta}$

$$= (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)$$

$$= i2\sin\theta$$

$\therefore Re(z) = 0$, $Im(z) = 2\sin\theta$

Q3) $z = e^{i2\theta} \cdot e^{-i\theta}$

$$= e^{i\theta}$$

$$= \cos\theta + i\sin\theta$$

$\therefore Re(z) = \cos\theta$, $Im(z) = \sin\theta$

Q4) $z = \frac{2e^{i\theta}(e^{i\theta}-1)}{3e^{i\theta}-1} = \frac{2e^{i\theta}(e^{i\theta}-1)\cdot(3e^{-i\theta}-1)}{(3e^{i\theta}-1)\cdot(3e^{-i\theta}-1)}$

$$= \frac{2(e^{i\theta}-1)\cdot(3-e^{i\theta})}{9-3e^{i\theta}-3e^{-i\theta}+1}$$

$$= \frac{2(3e^{i\theta}-3-e^{i2\theta}+e^{i\theta})}{10-3(\cos\theta+i\sin\theta)-3(\cos\theta-i\sin\theta)}$$

$$= \frac{2[4e^{i\theta}-e^{i2\theta}-3]}{10-6\cos\theta}$$

$$= \frac{4(\cos\theta + i\sin\theta) - (\cos 2\theta + i\sin 2\theta) - 3}{5 - 3\cos\theta}$$

$$= \frac{(4\cos\theta - \cos 2\theta - 3) + i(4\sin\theta - \sin 2\theta)}{5 - 3\cos\theta}$$

$$\therefore Re(z) = \frac{4\cos\theta - \cos 2\theta - 3}{5 - 3\cos\theta}$$

$$Im(z) = \frac{4\sin\theta - \sin 2\theta}{5 - 3\cos\theta}$$

Ex 2.2

$$y_{n+2} - 2a\, y_{n+1} + a^2 y_n = 0 \qquad (\star)$$

$$y_n = c_1 a^n + c_2 n a^n \qquad (\triangle)$$

Proof: Substitute $(\triangle)$ into $(\star)$

$$[c_1 a^{n+2} + c_2 (n+2) a^{n+2}] - 2a[c_1 a^{n+1} + c_2 (n+1) a^{n+1}] + a^2 [c_1 a^n + c_2 n a^n]$$

$$= c_1 [a^{n+2} - 2a \cdot a^{n+1} + a^2 \cdot a^n]$$
$$+ c_2 [(n+2) a^{n+2} - 2a(n+1) a^{n+1} + a^2 \cdot n a^n]$$

$$= c_1 [a^{n+2} - 2a^{n+2} + a^{n+2}]$$
$$+ c_2 \cdot a^{n+2} \cdot [(n+2) - 2(n+1) + n]$$

$$= 0$$

Q.E.D.

Ex 2.3

Q1) $y_{j+1} = y_{j-1}$

$\Rightarrow \quad y_{j+1} - y_{j-1} = 0$

$\Rightarrow \quad (E^2 - 1)\, y_{j-1} = 0$

$\therefore$ The characteristic equation is

$$\xi^2 - 1 = 0$$

$\Rightarrow \quad \xi_{1,2} = \pm 1$

$\Rightarrow \quad y_j = c_1 (1)^j + c_2(-1)^j = c_1 + c_2(-1)^j$

Q2) $y_{j+1} = 4y_j - 3y_{j-1}$

$\Rightarrow \quad y_{j+1} - 4y_j + 3y_{j-1} = 0$

$\Rightarrow \quad (E^2 - 4E + 3)\, y_{j-1} = 0$

$\Rightarrow \quad \xi^2 - 4\xi + 3 = 0$

$\Rightarrow \quad (\xi - 1)(\xi - 3) = 0$

$\Rightarrow \quad \xi_1 = 1, \ \xi_2 = 3$

$\Rightarrow \quad y_j = c_1 (1)^j + c_2 (3)^j$

$\Rightarrow \quad y_j = c_1 + c_2 (3)^j$

Q3) $y_{j+1} = 2y_{j-1} - y_j$

$\Rightarrow \quad y_{j+1} + y_j - 2y_{j-1} = 0$

$\Rightarrow \quad (E^2 + E - 2)\, y_j = 0$

$\Rightarrow \quad \xi^2 + \xi - 2 = 0$

$\Rightarrow \quad (\xi + 2)(\xi - 1) = 0$

$\Rightarrow \quad \xi_1 = -2, \ \xi_2 = 1$

$\Rightarrow \quad y_j = c_1 (-2)^j + c_2 (1)^j$

$\Rightarrow \quad y_j = c_1 (-2)^j + c_2$

Q4) $y_{j+1} + 9y_j - 9y_{j-1} - y_{j-2} = 0$

$\Rightarrow \quad (E^3 + 9E^2 - 9E - 1)\, y_{j-2} = 0$

$\Rightarrow \quad \xi^3 + 9\xi^2 - 9\xi - 1 = 0$

$\Rightarrow \quad (\xi^3 - 1) + (9\xi^2 - 9\xi) = 0$

$\Rightarrow (\xi-1)(\xi^2+\xi+1)+9\xi(\xi-1)=0$

$\Rightarrow (\xi-1)(\xi^2+10\xi+1)=0$

$\Rightarrow \xi_1=1,\ \xi_{2,3}=\dfrac{-10\pm\sqrt{10^2-4(1)(1)}}{2} = -5\pm2\sqrt{6}$

$\Rightarrow y_j = C_1(1)^j + C_2(-5+2\sqrt{6})^j + C_3(-5-2\sqrt{6})^j$

$\Rightarrow y_j = C_1 + C_2(-5+2\sqrt{6})^j + C_3(-5-2\sqrt{6})^j$

Ex **2.4**

Q1) $y_{n+2} - 9y_{n+1} + 20y_n = 0$

$\Rightarrow (E^2-9E+20)y_n=0$

$\Rightarrow \xi^2-9\xi+20=0$

$\Rightarrow (\xi-4)(\xi-5)=0$

$\Rightarrow \xi_1=4,\ \xi_2=5$

$\Rightarrow y_n = C_1(4)^n + C_2(5)^n$

Q2) $y_{n+2}+y_n+y_{n-1}=0$

$\Rightarrow (E^2+E+1)y_{n-1}=0$

$\Rightarrow \xi^2+\xi+1=0$

$\Rightarrow \xi_{1,2}=\dfrac{-1\pm\sqrt{1^2-4(1)(1)}}{2} = \dfrac{-1\pm\sqrt{3}i}{2}$

$\Rightarrow y_n = C_1\left(\dfrac{-1+\sqrt{3}i}{2}\right)^n + C_2\left(\dfrac{-1-\sqrt{3}i}{2}\right)^n$

Q3) $y_{n+2} = \dfrac{y_{n+1}+y_{n-1}}{2}$

$\Rightarrow 2y_{n+2} - y_{n+1} + 0y_n - y_{n-1} = 0$

$\Rightarrow (2E^3 - E^2 + 0E - 1)y_{n-1} = 0$

$\Rightarrow 2\xi^3 - \xi^2 - 1 = 0$

$\Rightarrow (\xi^3-\xi^2)+(\xi^3-1)=0$

$\Rightarrow \xi^2(\xi-1)+(\xi-1)(\xi^2+\xi+1)=0$

$\Rightarrow (\xi-1)(2\xi^2+\xi+1)=0$

$\Rightarrow \xi_1=1,\ \xi_{2,3}=\dfrac{-1\pm\sqrt{1^2-4(2)(1)}}{2(2)} = \dfrac{-1\pm\sqrt{7}i}{4}$

$$\Rightarrow y_n = C_1(1)^n + C_2\left(\frac{-1+\sqrt{7}i}{4}\right)^n + C_3\left(\frac{-1-\sqrt{7}i}{4}\right)^n$$

$$\Rightarrow y_n = C_1 + C_2\left(\frac{-1+\sqrt{7}i}{4}\right)^n + C_3\left(\frac{-1-\sqrt{7}i}{4}\right)^n$$

Q4) $y_{n+2} = -\dfrac{y_{n+1} + y_{n-1}}{2}$

$\Rightarrow 2y_{n+2} + y_{n+1} + 0y_n + y_{n-1} = 0$

$\Rightarrow (2E^3 + E^2 + 0E + 1)y_{n-1} = 0$

$\Rightarrow 2\xi^3 + \xi^2 + 1 = 0$

$\Rightarrow (\xi^3 + \xi^2) + (\xi^3 + 1) = 0$

$\Rightarrow \xi^2(\xi+1) + (\xi+1)(\xi^2 - \xi + 1) = 0$

$\Rightarrow (\xi+1)(2\xi^2 - \xi + 1) = 0$

$\Rightarrow \xi_1 = -1, \quad \xi_{2,3} = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(1)}}{2(2)} = \dfrac{1 \pm \sqrt{7}i}{4}$

$$\Rightarrow y_n = C_1(-1)^n + C_2\left(\frac{1+\sqrt{7}i}{4}\right)^n + C_3\left(\frac{1-\sqrt{7}i}{4}\right)^n$$

Q5) $y_{n+4} - 16y_n = 0$

$\Rightarrow (E^4 - 16)y_n = 0$

$\Rightarrow \xi^4 - 16 = 0$

$\Rightarrow (\xi^2 - 4)(\xi^2 + 4) = 0$

$\Rightarrow (\xi - 2)(\xi + 2)(\xi + 2i)(\xi - 2i) = 0$

$\Rightarrow \xi_{1,2} = \pm 2, \quad \xi_{3,4} = \pm 2i = 2e^{\pm i\frac{\pi}{2}}$

$\Rightarrow y_n = C_1(2)^n + C_2(-2)^n + 2^n\left(C_3 \cos\frac{n\pi}{2} + C_4 \sin\frac{n\pi}{2}\right)$

Q6) $y_{n+4} + 16y_n = 0$

$\Rightarrow (E^4 + 16)y_n = 0$

$\Rightarrow \xi^4 + 16 = 0$

$\Rightarrow \xi^4 = -16 = 16e^{i\pi}$

Let $\xi = re^{i\theta}$, $r > 0$, $\theta \in [0, 2\pi]$

$$\Rightarrow \begin{cases} \xi^4 = r^4 e^{i4\theta} \\ \xi^4 = 16e^{i\pi} \end{cases}$$

$$\Rightarrow \begin{cases} r^4 = 16 \\ 4\theta = (2k+1)\pi \end{cases} \Rightarrow \theta = \frac{2k+1}{4}\pi \quad , k \in \mathbb{Z}$$

$\Rightarrow \quad r = 2$

$$\left.\begin{array}{ll} k=0, & \theta = \frac{1}{4}\pi \\ k=1, & \theta = \frac{3}{4}\pi \\ k=2, & \theta = \frac{5}{4}\pi \\ k=3, & \theta = \frac{7}{4}\pi \end{array}\right\} \left.\begin{array}{l} \Rightarrow \xi_1 = \sqrt{2} + \sqrt{2}i \\ \Rightarrow \xi_3 = -\sqrt{2} + \sqrt{2}i \\ \Rightarrow \xi_4 = -\sqrt{2} - \sqrt{2}i \\ \Rightarrow \xi_2 = \sqrt{2} - \sqrt{2}i \end{array}\right\} \Rightarrow \begin{cases} \xi_{1,2} = \sqrt{2} \pm \sqrt{2}i \\ \xi_{3,4} = -\sqrt{2} \pm \sqrt{2}i \end{cases}$$

$$k=4, \quad \theta = \frac{9}{4}\pi = 2\pi + \frac{1}{4}\pi$$

$$\Rightarrow \xi_{1,2} = 2e^{\pm i\frac{\pi}{4}}, \quad \xi_{3,4} = 2e^{\pm i\frac{3\pi}{4}}$$

$$\Rightarrow y_n = 2^n\left(c_1\cos\frac{n\pi}{4} + c_2\sin\frac{n\pi}{4}\right) + 2^n\left(c_3\cos\frac{3n\pi}{4} + c_4\sin\frac{3n\pi}{4}\right)$$

Ex 2.5

$$y_n = y_{n-1} + y_{n-2}, \quad y_0 = 0, \quad y_1 = 1$$

$$\Rightarrow \quad y_n - y_{n-1} - y_{n-2} = 0$$

$$\Rightarrow \quad (E^2 - E - 1)y_{n-2} = 0$$

$$\Rightarrow \quad \xi^2 - \xi - 1 = 0$$

$$\Rightarrow \quad \xi_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad y_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\begin{cases} y_0 = c_1 + c_2 = 0 \\ y_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases}$$

$$\Rightarrow c_1\left(\frac{1+\sqrt{5}}{2}\right) - c_1\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\Rightarrow c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = -\frac{1}{\sqrt{5}}$$

$$\Rightarrow y_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$