Q1(a) $y_{j+1} = y_j + \frac{h}{2}(3f_j - f_{j-1})$

$\Rightarrow -y_j + y_{j+1} = h(-\frac{1}{2}f_{j-1} + \frac{3}{2}f_j)$

$\Leftrightarrow 0y_{j-1} - y_j + y_{j+1} = h(-\frac{1}{2}f_{j-1} + \frac{3}{2}f_j + 0f_{j+1})$

$\Rightarrow \alpha_0 = 0, \alpha_1 = -1, \alpha_2 = 1, \quad \beta_0 = -\frac{1}{2}, \beta_1 = \frac{3}{2}, \beta_2 = 0$

$\rho(\xi) = \sum_{i=0}^{K} \alpha_i \xi^i = -\xi + \xi^2 = \xi(\xi - 1)$

Let $\rho(\xi) = \xi(\xi-1) = 0$

$\Rightarrow \xi_1 = 0, \quad \xi_2 = 1$

This satisfies the root condition $|\xi_i| \leq 1$, so the method is stable. Besides, there is only one root on the unit circle ($\xi_2 = 1$), so the method is stronly stable.

Q1(b) $y_{j+1} + 9y_j - 9y_{j-1} - y_{j-2} = 6h(f_j + f_{j-1})$

$\Rightarrow -y_{j-2} - 9y_{j-1} + 9y_j + y_{j+1} = h(0f_{j-2} + 6f_{j-1} + 6f_j + 0f_{j+1})$

$\Rightarrow \alpha_0 = -1, \alpha_1 = -9, \alpha_2 = 9, \alpha_3 = 1, \beta_0 = 0, \beta_1 = 6, \beta_2 = 6, \beta_3 = 0$

$\therefore \rho(\xi) = \sum_{i=0}^{K} \alpha_i \xi^i = \xi^3 + 9\xi^2 - 9\xi - 1$

$= (\xi^3 - 1) + (9\xi^2 - 9\xi)$

$= (\xi - 1)(\xi^2 + \xi + 1) + 9\xi(\xi - 1)$

$= (\xi - 1)(\xi^2 + 10\xi + 1)$

Let $\rho(\xi) = (\xi - 1)(\xi^2 + 10\xi + 1) = 0$

$\Rightarrow \xi_1 = 1$ and $\xi^2 + 10\xi + 1 = 0$

$\xi^2 + 10\xi + 1 = 0$

$\Rightarrow \xi^2 + 2(5)\xi + 25 = 24$

$\Rightarrow (\xi + 5)^2 = 24$

$\Rightarrow \xi_{2,3} = -5 \pm 2\sqrt{6}$, $\xi_2 \approx -0.1$, $\xi_3 \approx -9.9$

$|\xi_1| \leq 1$, $|\xi_2| \leq 1$, but $|\xi_3| > 1$

This method doesn't satisfy the root condition, so it is not stable.

Q1(c) $y_{j+1} = 4y_j - 3y_{j-1} - 2hf_{j-1}$

$\Rightarrow 3y_{j-1} - 4y_j + y_{j+1} = -2hf_{j-1} + 0f_j + 0f_{j+1}$

$\Rightarrow \alpha_0 = 3, \alpha_1 = -4, \alpha_2 = 1, \beta_0 = -2, \beta_1 = 0, \beta_2 = 0$

$\Rightarrow \rho(\xi) = \sum_{i=0}^{k} \alpha_i \xi^i = \xi^2 - 4\xi + 3 = (\xi - 1)(\xi - 3)$

Let $\rho(\xi) = (\xi - 1)(\xi - 3) = 0$

$\Rightarrow \xi_1 = 1, \ \xi_2 = 3$

This method doesn't satisfy the root condition, so it is not stable.

Q1 (d) $y_{j+1} = 2y_{j-1} - y_j + \frac{h}{2}(5f_j + f_{j-1})$

$\Rightarrow -2y_{j-1} + y_j + y_{j+1} = h\left(\frac{1}{2}f_{j-1} + \frac{5}{2}f_j + 0f_{j+1}\right)$

$\Rightarrow \alpha_0 = -2, \alpha_1 = 1, \alpha_2 = 1, \beta_0 = \frac{1}{2}, \beta_1 = \frac{5}{2}, \beta_2 = 0$

$\Rightarrow \rho(\xi) = \sum_{i=0}^{K} \alpha_i \xi^i = \xi^2 + \xi - 2 = (\xi - 1)(\xi + 2) = 0$

$\Rightarrow \xi_1 = 1, \xi_2 = -2$

This method doesn't satisfy the root condition, it is not stable.

2 (a) $y_{j+1} = y_j + \frac{h}{2}(3f_j - f_{j-1})$

From Q1 (a), we know that

$\alpha_0 = 0, \alpha_1 = -1, \alpha_2 = 1, \beta_0 = -\frac{1}{2}, \beta_1 = \frac{3}{2}, \beta_2 = 0$

Use the formulae in Chapter 3 to calculate C_p, so

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 = 0 + (-1) + 1 = 0$

$C_1 = (\alpha_1 + 2\alpha_2) - (\beta_0 + \beta_1 + \beta_2)$

$= (-1 + 2) - \left(-\frac{1}{2} + \frac{3}{2} + 0\right)$

$= 0$

$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2) - \frac{1}{1!}(\beta_1 + 2\beta_2)$

$= \frac{1}{2}(-1 + 4) - \left(\frac{3}{2} + 0\right)$

$= 0$

$$C_3 = \frac{1}{3!}(\alpha_1 + 2^3\alpha_2) - \frac{1}{2!}(\beta_1 + 2^2\beta_2)$$
$$= \frac{1}{6}(-1+8) - \frac{1}{2}\left(\frac{3}{2}+0\right)$$
$$= \frac{7}{6} - \frac{3}{4}$$
$$= \frac{5}{12}$$

$\therefore$ The method is of 2nd order accuracy, the error constant in $\frac{5}{12}$.

Q 2(b) $y_{j+1} + 9y_j - 9y_{j-1} - y_{j-2} = 6h(f_j + f_{j-1})$

From Q 1(b), we know that

$\alpha_0 = -1, \alpha_1 = -9, \alpha_2 = 9, \alpha_3 = 1, \beta_0 = 0, \beta_1 = 6, \beta_2 = 6, \beta_3 = 0$

$\Rightarrow C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = -1 + 9 + 9 + 1 = 0$

$$C_1 = (\alpha_1 + 2\alpha_2 + 3\alpha_3) - (\beta_0 + \beta_1 + \beta_2 + \beta_3)$$
$$= [-9 + 2(9) + 3(1)] - [0 + 6 + 6 + 0]$$
$$= 12 - 12 = 0$$

$$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3) - \frac{1}{1!}(\beta_1 + 2\beta_2 + 3\beta_3)$$
$$= \frac{1}{2}[-9 + 4(9) + 9(1)] - [6 + 2(6) + 3(0)]$$
$$= 18 - 18 = 0$$

$$C_3 = \frac{1}{3!}(\alpha_1 + 2^3\alpha_2 + 3^3\alpha_3) - \frac{1}{2!}(\beta_1 + 2^2\beta_2 + 3^2\beta_3)$$
$$= \frac{1}{6}[-9 + 8(9) + 27(1)] - \frac{1}{2}[6 + 4(6) + 9(0)]$$
$$= 15 - 15 = 0$$

$$C_4 = \frac{1}{4!}(\alpha_1 + 2^4\alpha_2 + 3^4\alpha_3) - \frac{1}{3!}(\beta_1 + 2^3\beta_2 + 3^3\beta_3)$$
$$= \frac{1}{24}[-9 + 16(9) + 81(1)] - \frac{1}{6}[6 + 8(6) + 27(0)]$$
$$= 9 - 9 = 0$$
$$C_5 = \frac{1}{5!}(\alpha_1 + 2^5\alpha_2 + 3^5\alpha_3) - \frac{1}{4!}(\beta_1 + 2^4\beta_2 + 3^4\beta_3)$$
$$= \frac{1}{120}[-9 + 32(9) + 243(1)] - \frac{1}{24}[6 + 16(6) + 81(0)]$$
$$= \frac{87}{20} - \frac{17}{4}$$
$$= \frac{1}{10}$$
$\therefore$ The method is of 4th order accuracy,
the error constant is in $\frac{1}{10}$.

Q2(c) $y_{j+1} = 4y_j - 3y_{j-1} - 2hf_{j-1}$
$\Rightarrow 3y_{j-1} - 4y_j + y_{j+1} = -2hf_{j-1} + 0f_j + 0f_{j+1}$
$\Rightarrow \alpha_0 = 3, \alpha_1 = -4, \alpha_2 = 1, \beta_0 = -2, \beta_1 = 0, \beta_2 = 0$
$\Rightarrow C_0 = \alpha_0 + \alpha_1 + \alpha_2 = 3 - 4 + 1 = 0$
$$C_1 = (\alpha_1 + 2\alpha_2) - (\beta_0 + \beta_1 + \beta_2)$$
$$= (-4 + 2) - (-2 + 0 + 0)$$
$$= 0$$
$$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2) - \frac{1}{1!}(\beta_1 + 2\beta_2)$$
$$= \frac{1}{2}(-4 + 4) - 0$$
$$= 0$$

$$C_3 = \frac{1}{3!}(\alpha_1 + 2^3\alpha_2) - \frac{1}{2!}(\beta_1 + 2^2\beta_2)$$
$$= \frac{1}{6}(-4+8) - 0$$
$$= \frac{2}{3}$$

$\therefore$ The method is of 2nd order accuracy, the error constant in $\frac{2}{3}$.

Q2 (d) $y_{j+1} = 2y_{j-1} - y_j + \frac{h}{2}(5f_j + f_{j-1})$

$\Rightarrow -2y_{j-1} + y_j + y_{j+1} = h\left(\frac{1}{2}f_{j-1} + \frac{5}{2}f_j + 0f_{j+1}\right)$

$\Rightarrow \alpha_0 = -2,\ \alpha_1 = 1,\ \alpha_2 = 1,\ \beta_0 = \frac{1}{2},\ \beta_1 = \frac{5}{2},\ \beta_2 = 0$

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 = -2 + 1 + 1 = 0$$

$$C_1 = (\alpha_1 + 2\alpha_2) - (\beta_0 + \beta_1 + \beta_2)$$
$$= (1+2) - \left(\frac{1}{2} + \frac{5}{2} + 0\right)$$
$$= 0$$

$$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2) - \frac{1}{1!}(\beta_1 + 2\beta_2)$$
$$= \frac{1}{2}(1+4) - \left(\frac{5}{2} + 0\right)$$
$$= 0$$

$$C_3 = \frac{1}{3!}(\alpha_1 + 2^3\alpha_2) - \frac{1}{2!}(\beta_1 + 2^2\beta_2)$$
$$= \frac{1}{6}(1+8) - \frac{1}{2}\left(\frac{5}{2}\right) = \frac{1}{4}$$

$\therefore$ The method is of 2nd order accuracy, the error constant in $\frac{1}{4}$.