Ex 3.1

Q1) $y_{j+2} = y_j + \frac{h}{3}(f_{j+2} + 4f_{j+1} + f_j)$ G-N forward 0 to 2

$$\frac{dy}{dx} = f$$

$$\Rightarrow dy = f\,dx$$

$$\Rightarrow \int_{x_j}^{x_{j+2}} dy = \int_{x_j}^{x_{j+2}} f\,dx$$

$$\Rightarrow y_{j+2} - y_j = h\int_0^2 f\,ds$$

$$= h\int_0^2 (f_j + s\Delta f_j + \frac{s(s-1)}{2!}\Delta^2 f_j)\,ds$$

$$= h\left[f_j s + \tfrac{1}{2}\Delta f_j s^2 + \Delta^2 f_j(\tfrac{s^3}{6} - \tfrac{s^2}{4})\right]_0^{s=2}$$

$$= h\{[f_j(2) + \tfrac{1}{2}\Delta f_j (2)^2 + \Delta^2 f_j(\tfrac{2^3}{6} - \tfrac{2^2}{4})] - 0\}$$

$$= h(2f_j + 2\Delta f_j + \tfrac{1}{3}\Delta^2 f_j)$$

$$= h[2f_j + 2f_{j+1} - 2f_j + \tfrac{1}{3}(f_{j+2} - 2f_{j+1} + f_j)]$$

$$= \frac{h}{3}(f_{j+2} + 4f_{j+1} + f_j)$$

$$\Rightarrow y_{j+2} = y_j + \frac{h}{3}(f_{j+2} + 4f_{j+1} + f_j)$$

Q2) $y_{j+2} = y_{j+1} + \frac{h}{12}(5f_{j+2} + 8f_{j+1} - f_j)$ G-N forward 1 to 2

$$\int_{x_{j+1}}^{x_{j+2}} dy = \int_{x_{j+1}}^{x_{j+2}} f\,dx$$

$$\Rightarrow y_{j+2} - y_{j+1} = h\int_1^2 f\,ds$$

$$= h\int_1^2 (f_j + s\Delta f_j + \frac{s(s-1)}{2}\Delta^2 f_j)\,ds$$

$$= h\left[f_j s + \frac{\Delta f_j}{2}s^2 + \Delta^2 f_j(\frac{s^3}{6} - \frac{s^2}{4})\right]_1^2$$

$$= h\{f_j(2-1) + \frac{\Delta f_j}{2}(2^2 - 1^2) + \Delta^2 f_j[(\frac{2^3}{6} - \frac{2^2}{4}) - (\frac{1^3}{6} - \frac{1^2}{4})]\}$$

$$= h\{f_j + \tfrac{3}{2}\Delta f_j + \tfrac{5}{12}\Delta^2 f_j\}$$

$$= h\{f_j + \tfrac{3}{2}(f_{j+1} - f_j) + \tfrac{5}{12}(f_{j+2} - 2f_{j+1} + f_j)\}$$

$$= h\{f_j(1 - \tfrac{3}{2} + \tfrac{5}{12}) + f_{j+1}(\tfrac{3}{2} - \tfrac{5}{6}) + \tfrac{5}{12}f_{j+2}\}$$

$$= h\{-\tfrac{1}{12}f_j + \tfrac{2}{3}f_{j+1} + \tfrac{5}{12}f_{j+2}\}$$

$$= \frac{h}{12}(5f_{j+2} + 8f_{j+1} - f_j)$$

$$\Rightarrow y_{j+2} = y_{j+1} + \frac{h}{12}(5f_{j+2} + 8f_{j+1} - f_j)$$

Q3) $y_{j+3} = y_{j-1} + \frac{h}{3}(8f_{j+2} - 4f_{j+1} + 8f_j)$ G-N forward −1 to 3

$\int_{x_{j-1}}^{x_{j+3}} dy = \int_{x_{j-1}}^{x_{j+3}} f\,dx$

$$\Rightarrow y_{j+3} - y_{j-1} = h\int_{-1}^{3}\left(f_j + s\Delta f_j + \frac{s(s-1)}{2!}\Delta^2 f_j\right) ds$$
$$= h\left[f_j s + \tfrac{1}{2}\Delta f_j s^2 + \Delta^2 f_j\left(\frac{s^3}{6} - \frac{s^2}{4}\right)\right]_{-1}^{3}$$
$$= h\left\{f_j[3-(-1)] + \tfrac{1}{2}\Delta f_j[3^2-(-1)^2] + \Delta^2 f_j\left[\left(\frac{3^3}{6} - \frac{3^2}{4}\right) - \left(\frac{(-1)^3}{6} - \frac{(-1)^2}{4}\right)\right]\right\}$$
$$= h\left\{4f_j + 4\Delta f_j + \frac{8}{3}\Delta^2 f_j\right\}$$
$$= h\left\{4f_j + 4(f_{j+1} - f_j) + \frac{8}{3}(f_{j+2} - 2f_{j+1} + f_j)\right\}$$
$$= h\left\{\frac{8}{3}f_{j+2} - \frac{4}{3}f_{j+1} + \frac{8}{3}f_j\right\}$$
$$= \frac{h}{3}(8f_{j+2} - 4f_{j+1} + 8f_j\}$$
$$\Rightarrow y_{j+3} = y_{j-1} + \frac{h}{3}(8f_{j+2} - 4f_{j+1} + 8f_j)$$

Q4) $y_j = y_{j-1} + \frac{h}{12}(5f_j + 8f_{j-1} - f_{j-2})$ G-N backward 0 to −1

$\int_{x_j}^{x_{j-1}} dy = \int_{x_j}^{x_{j-1}} f\,dx$

$\Rightarrow \int_{x_{j-1}}^{x_j} dy = \int_{x_{j-1}}^{x_j} f\,dx$

$$\Rightarrow y_j - y_{j-1} = h\int_{-1}^{0} f\,ds$$
$$= h\int_{-1}^{0}\left(f_j + s\nabla f_j + \frac{s(s+1)}{2!}\nabla^2 f_j\right) ds$$
$$= h\left[f_j s + \tfrac{1}{2}\nabla f_j s^2 + \nabla^2 f_j\left(\frac{s^3}{6} + \frac{s^2}{4}\right)\right]_{-1}^{0}$$
$$= h\left\{f_j[0-(-1)] + \tfrac{1}{2}\nabla f_j[0^2-(-1)^2] + \nabla^2 f_j\left[\frac{0^3-(-1)^3}{6} + \frac{0^2-(-1)^2}{4}\right]\right\}$$
$$= h\left\{f_j - \tfrac{1}{2}\nabla f_j - \tfrac{1}{12}\nabla^2 f_j\right\}$$
$$= h\left\{f_j - \tfrac{1}{2}(f_j - f_{j-1}) - \tfrac{1}{12}(f_j - 2f_{j-1} + f_{j-2})\right\}$$
$$= h\left\{\frac{5}{12}f_j + \frac{8}{12}f_{j-1} - \frac{1}{12}f_{j-2}\right\}$$
$$\Rightarrow y_j = y_{j-1} + \frac{h}{12}(5f_j + 8f_{j-1} - f_{j-2})$$