

Ex 2.1

Q1)

$$\begin{aligned}z &= \frac{4+3i}{3-4i} = \frac{(4+3i)(3+4i)}{(3-4i)(3+4i)} \\&= \frac{12+16i+9i+12i^2}{9-16i^2} \\&= \frac{25i}{25} \\&= i\end{aligned}$$

$$\therefore \operatorname{Re}(z) = 0, \operatorname{Im}(z) = 1$$

Q2)

$$\begin{aligned}z &= e^{i\theta} - e^{-i\theta} \\&= (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) \\&= i2\sin\theta\end{aligned}$$

$$\therefore \operatorname{Re}(z) = 0, \operatorname{Im}(z) = 2\sin\theta$$

Q3)

$$\begin{aligned}z &= e^{i2\theta} \cdot e^{-i\theta} \\&= e^{i\theta} \\&= \cos\theta + i\sin\theta\end{aligned}$$

$$\therefore \operatorname{Re}(z) = \cos\theta, \operatorname{Im}(z) = \sin\theta$$

Q4)

$$\begin{aligned}z &= \frac{2e^{i\theta}(e^{i\theta}-1)}{3e^{i\theta}-1} = \frac{2e^{i\theta}(e^{i\theta}-1)(3e^{-i\theta}-1)}{(3e^{i\theta}-1)(3e^{-i\theta}-1)} \\&= \frac{2(e^{i\theta}-1)(3-e^{i\theta})}{9-3e^{i\theta}-3e^{-i\theta}+1} \\&= \frac{2(3e^{i\theta}-3-e^{i2\theta}+e^{i\theta})}{10-3(\cos\theta+i\sin\theta)-3(\cos\theta-i\sin\theta)} \\&= \frac{2[4e^{i\theta}-e^{i2\theta}-3]}{10-6\cos\theta}\end{aligned}$$

$$= \frac{4(\cos\theta + i\sin\theta) - (\cos 2\theta + i\sin 2\theta) - 3}{5 - 3\cos\theta}$$

$$= \frac{(4\cos\theta - \cos 2\theta - 3) + i(4\sin\theta - \sin 2\theta)}{5 - 3\cos\theta}$$

$$\therefore \operatorname{Re}(z) = \frac{4\cos\theta - \cos 2\theta - 3}{5 - 3\cos\theta}$$

$$\operatorname{Im}(z) = \frac{4\sin\theta - \sin 2\theta}{5 - 3\cos\theta}$$

Ex 2.2

$$y_{n+2} - 2ay_{n+1} + a^2y_n = 0 \quad (*)$$

$$y_n = c_1a^n + c_2na^n \quad (\Delta)$$

Proof: Substitute (Δ) into $(*)$

$$\begin{aligned} & [c_1a^{n+2} + c_2(n+2)a^{n+2}] - 2a[c_1a^{n+1} + c_2(n+1)a^{n+1}] + a^2[c_1a^n + c_2na^n] \\ = & c_1[a^{n+2} - 2a \cdot a^{n+1} + a^2 \cdot a^n] \\ & + c_2[(n+2)a^{n+2} - 2a(n+1)a^{n+1} + a^2 \cdot na^n] \\ = & c_1[a^{n+2} - 2a^{n+2} + a^{n+2}] \\ & + c_2 \cdot a^{n+2} \cdot [(n+2) - 2(n+1) + n] \\ = & 0 \end{aligned}$$

Q.E.D.

Ex 2.3

$$Q1) \quad y_{j+1} = y_{j-1}$$

$$\Rightarrow y_{j+1} - y_{j-1} = 0$$

$$\Rightarrow (E^2 - 1)y_{j-1} = 0$$

\therefore The characteristic equation is

$$\xi^2 - 1 = 0$$

$$\Rightarrow \xi_{1,2} = \pm 1$$

$$\Rightarrow y_j = C_1(1)^j + C_2(-1)^j = C_1 + C_2(-1)^j$$

$$Q2) \quad y_{j+1} = 4y_j - 3y_{j-1}$$

$$\Rightarrow y_{j+1} - 4y_j + 3y_{j-1} = 0$$

$$\Rightarrow (E^2 - 4E + 3)y_{j-1} = 0$$

$$\Rightarrow \xi^2 - 4\xi + 3 = 0$$

$$\Rightarrow (\xi - 1)(\xi - 3) = 0$$

$$\Rightarrow \xi_1 = 1, \xi_2 = 3$$

$$\Rightarrow y_j = C_1(1)^j + C_2(3)^j$$

$$\Rightarrow y_j = C_1 + C_2(3)^j$$

$$Q3) \quad y_{j+1} = 2y_{j-1} - y_j$$

$$\Rightarrow y_{j+1} + y_j - 2y_{j-1} = 0$$

$$\Rightarrow (E^2 + E - 2)y_j = 0$$

$$\Rightarrow \xi^2 + \xi - 2 = 0$$

$$\Rightarrow (\xi + 2)(\xi - 1) = 0$$

$$\Rightarrow \xi_1 = -2, \xi_2 = 1$$

$$\Rightarrow y_j = C_1(-2)^j + C_2(1)^j$$

$$\Rightarrow y_j = C_1(-2)^j + C_2$$

$$Q4) \quad y_{j+1} + 9y_j - 9y_{j-1} - y_{j-2} = 0$$

$$\Rightarrow (E^3 + 9E^2 - 9E - 1)y_{j-2} = 0$$

$$\Rightarrow \xi^3 + 9\xi^2 - 9\xi - 1 = 0$$

$$\Rightarrow (\xi^3 - 1) + (9\xi^2 - 9\xi) = 0$$

$$\Rightarrow (\xi-1)(\xi^2+\xi+1)+9\xi(\xi-1)=0$$

$$\Rightarrow (\xi-1)(\xi^2+10\xi+1)=0$$

$$\Rightarrow \xi_1=1, \xi_{2,3}=\frac{-10\pm\sqrt{10^2-4(1)(1)}}{2}=-5\pm 2\sqrt{6}$$

$$\Rightarrow y_j=C_1(1)^j+C_2(-5+2\sqrt{6})^j+C_3(-5-2\sqrt{6})^j$$

$$\Rightarrow y_j=C_1+C_2(-5+2\sqrt{6})^j+C_3(-5-2\sqrt{6})^j$$

Ex 2.4

$$Q1) \quad y_{n+2}-9y_{n+1}+20y_n=0$$

$$\Rightarrow (E^2-9E+20)y_n=0$$

$$\Rightarrow \xi^2-9\xi+20=0$$

$$\Rightarrow (\xi-4)(\xi-5)=0$$

$$\Rightarrow \xi_1=4, \xi_2=5$$

$$\Rightarrow y_n=C_1(4)^n+C_2(5)^n$$

$$Q2) \quad y_{n+2}+y_{n+1}+y_n=0$$

$$\Rightarrow (E^2+E+1)y_n=0$$

$$\Rightarrow \xi^2+\xi+1=0$$

$$\Rightarrow \xi_{1,2}=\frac{-1\pm\sqrt{1^2-4(1)(1)}}{2}=\frac{-1\pm\sqrt{3}i}{2}$$

$$\Rightarrow y_n=C_1\left(\frac{-1+\sqrt{3}i}{2}\right)^n+C_2\left(\frac{-1-\sqrt{3}i}{2}\right)^n$$

$$Q3) \quad y_{n+2}=\frac{y_{n+1}+y_{n-1}}{2}$$

$$\Rightarrow 2y_{n+2}-y_{n+1}+0y_n-y_{n-1}=0$$

$$\Rightarrow (2E^3-E^2+0E-1)y_{n-1}=0$$

$$\Rightarrow 2\xi^3-\xi^2-1=0$$

$$\Rightarrow (\xi^3-\xi^2)+(\xi^3-1)=0$$

$$\Rightarrow \xi^2(\xi-1)+(\xi-1)(\xi^2+\xi+1)=0$$

$$\Rightarrow (\xi-1)(2\xi^2+\xi+1)=0$$

$$\Rightarrow \xi_1=1, \xi_{2,3}=\frac{-1\pm\sqrt{1^2-4(2)(1)}}{2(2)}=\frac{-1\pm\sqrt{7}i}{4}$$

$$\Rightarrow y_n = C_1 (1)^n + C_2 \left(\frac{-1 + \sqrt{7}i}{4} \right)^n + C_3 \left(\frac{-1 - \sqrt{7}i}{4} \right)^n$$

$$\Rightarrow y_n = C_1 + C_2 \left(\frac{-1 + \sqrt{7}i}{4} \right)^n + C_3 \left(\frac{-1 - \sqrt{7}i}{4} \right)^n$$

$$Q4) \quad y_{n+2} = -\frac{y_{n+1} + y_{n-1}}{2}$$

$$\Rightarrow 2y_{n+2} + y_{n+1} + 0y_n + y_{n-1} = 0$$

$$\Rightarrow (2E^3 + E^2 + 0E + 1)y_{n-1} = 0$$

$$\Rightarrow 2\xi^3 + \xi^2 + 1 = 0$$

$$\Rightarrow (\xi^3 + \xi^2) + (\xi^3 + 1) = 0$$

$$\Rightarrow \xi^2(\xi + 1) + (\xi + 1)(\xi^2 - \xi + 1) = 0$$

$$\Rightarrow (\xi + 1)(2\xi^2 - \xi + 1) = 0$$

$$\Rightarrow \xi_1 = -1, \quad \xi_{2,3} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(1)}}{2(2)} = \frac{1 \pm \sqrt{7}i}{4}$$

$$\Rightarrow y_n = C_1 (-1)^n + C_2 \left(\frac{1 + \sqrt{7}i}{4} \right)^n + C_3 \left(\frac{1 - \sqrt{7}i}{4} \right)^n$$

$$Q5) \quad y_{n+4} - 16y_n = 0$$

$$\Rightarrow (E^4 - 16)y_n = 0$$

$$\Rightarrow \xi^4 - 16 = 0$$

$$\Rightarrow (\xi^2 - 4)(\xi^2 + 4) = 0$$

$$\Rightarrow (\xi - 2)(\xi + 2)(\xi + 2i)(\xi - 2i) = 0$$

$$\Rightarrow \xi_{1,2} = \pm 2, \quad \xi_{3,4} = \pm 2i = 2e^{\pm i\frac{\pi}{2}}$$

$$\Rightarrow y_n = C_1 (2)^n + C_2 (-2)^n + 2^n (C_3 \cos \frac{n\pi}{2} + C_4 \sin \frac{n\pi}{2})$$

$$Q6) \quad y_{n+4} + 16y_n = 0$$

$$\Rightarrow (E^4 + 16)y_n = 0$$

$$\Rightarrow \xi^4 + 16 = 0$$

$$\Rightarrow \xi^4 = -16 = 16e^{i\pi}$$

$$\text{Let } \xi = r e^{i\theta}, \quad r > 0, \quad \theta \in [0, 2\pi]$$

$$\Rightarrow \left. \begin{aligned} \xi^4 &= r^4 e^{i4\theta} \\ \xi^4 &= 16 e^{i\pi} \end{aligned} \right\}$$

$$\Rightarrow \begin{cases} r^4 = 16 \\ 4\theta = (2k+1)\pi \end{cases} \Rightarrow \theta = \frac{2k+1}{4}\pi, k \in \mathbb{Z}$$

$$\Rightarrow r = 2$$

$$\begin{cases} k=0, \theta = \frac{1}{4}\pi \Rightarrow \xi_1 = \sqrt{2} + \sqrt{2}i \\ k=1, \theta = \frac{3}{4}\pi \Rightarrow \xi_3 = -\sqrt{2} + \sqrt{2}i \\ k=2, \theta = \frac{5}{4}\pi \Rightarrow \xi_4 = -\sqrt{2} - \sqrt{2}i \\ k=3, \theta = \frac{7}{4}\pi \Rightarrow \xi_2 = \sqrt{2} - \sqrt{2}i \\ k=4, \theta = \frac{9}{4}\pi = 2\pi + \frac{1}{4}\pi \end{cases} \Rightarrow \begin{cases} \xi_{1,2} = \sqrt{2} \pm \sqrt{2}i \\ \xi_{3,4} = -\sqrt{2} \pm \sqrt{2}i \end{cases}$$

$$\Rightarrow \xi_{1,2} = 2e^{\pm i\frac{\pi}{4}}, \xi_{3,4} = 2e^{\pm i\frac{3\pi}{4}}$$

$$\Rightarrow y_n = 2^n (C_1 \cos \frac{n\pi}{4} + C_2 \sin \frac{n\pi}{4}) + 2^n (C_3 \cos \frac{3n\pi}{4} + C_4 \sin \frac{3n\pi}{4})$$

Ex 2.5

$$y_n = y_{n-1} + y_{n-2}, y_0 = 0, y_1 = 1$$

$$\Rightarrow y_n - y_{n-1} - y_{n-2} = 0$$

$$\Rightarrow (E^2 - E - 1)y_{n-2} = 0$$

$$\Rightarrow \xi^2 - \xi - 1 = 0$$

$$\Rightarrow \xi_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow y_n = C_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\begin{cases} y_0 = C_1 + C_2 = 0 \\ y_1 = C_1 \left(\frac{1+\sqrt{5}}{2}\right) + C_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases}$$

$$\Rightarrow C_1 \left(\frac{1+\sqrt{5}}{2}\right) - C_1 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\Rightarrow C_1 = \frac{1}{\sqrt{5}}, C_2 = -\frac{1}{\sqrt{5}}$$

$$\Rightarrow y_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$