Ex 4.1

Q1) $y_j = y_{j-1} + \frac{h}{12}(5f_j + 8f_{j-1} - f_{j-2})$ G.N Backward 0 to -1

$dy = f(x,y)\,dx$ $\boxed{dx = h\,ds}$

$\Rightarrow \int_{x_j}^{x_{j-1}} dy = \int_{x_j}^{x_{j-1}} f\,dx$

$\Rightarrow y_{j-1} - y_j = h\int_0^{-1} [f_j + s\nabla f_j + \frac{s(s+1)}{2!}\nabla^2 f_j + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_j]\,ds$

$= h\int_0^{-1} [f_j + s\nabla f_j + \frac{1}{2}\nabla^2 f_j\,(s^2+s)]\,ds + h\int_0^{-1}\frac{1}{6}(s^3+3s^2+2s)\nabla^3 f_j\,ds$

$= h\left[f_j s + \frac{1}{2}s^2\nabla f_j + \frac{1}{2}\nabla^2 f_j\left(\frac{s^3}{3}+\frac{s^2}{2}\right)\right]_0^{-1}$

$+ \frac{h}{6}\nabla^3 f_j\left[\frac{1}{4}s^4 + s^3 + s^2\right]_0^{-1}$

$= h[-f_j + \frac{1}{2}\nabla f_j + \frac{1}{12}\nabla^2 f_j] + \frac{h}{24}\nabla^3 f_j$

$= h[-f_j + \frac{1}{2}(f_j - f_{j-1}) + \frac{1}{12}(f_j - 2f_{j-1} + f_{j-2})] + \frac{h}{24}\nabla^3 f_j$

$= h[(-f_j + \frac{1}{2}f_j + \frac{1}{12}f_j) + (-\frac{1}{2}f_{j-1} - \frac{2}{12}f_{j-1}) + \frac{1}{12}f_{j-2}] + \frac{h}{24}\nabla^3 f_j$

$= h[-\frac{5}{12}f_j - \frac{8}{12}f_{j-1} + \frac{1}{12}f_{j-2}] + \frac{h}{24}\nabla^3 f_j$

$\Rightarrow y_{j-1} - y_j = h[-\frac{5}{12}f_j - \frac{8}{12}f_{j-1} + \frac{1}{12}f_{j-2}] + \frac{h}{24}\nabla^3 f_j$

$\Rightarrow y_j - y_{j-1} = \frac{h}{12}[5f_j + 8f_{j-1} - f_{j-2}] - \frac{h}{24}\nabla^3 f_j$

$\Rightarrow \varepsilon = -\frac{h}{24}\nabla^3 f_j = -\frac{h^4}{24}f'''(\xi)$ $x_{j-1} \le \xi \le x_j$

Q2) $y_{j+1} = y_j + \frac{h}{2}(f_{j+1} + f_j)$

We can use G-N forward 0 to 1

$dy = f\,dx$

$\Rightarrow \int_{x_j}^{x_{j+1}} dy = \int_{x_j}^{x_{j+1}} f\,dx$

$\Rightarrow y_{j+1} - y_j = h\int_0^1 [f_j + s\Delta f_j + \frac{s(s-1)}{2!}\Delta^2 f_j]\,ds$

$= h\int_0^1 [f_j + s\Delta f_j]\,ds + \frac{h}{2}\Delta^2 f_j\int_0^1 (s^2 - s)\,ds$

$= h[f_j s + \frac{1}{2}\Delta f_j s^2]_0^1 + \frac{h}{2}\Delta^2 f_j\,[\frac{1}{3}s^3 - \frac{1}{2}s^2]_0^1$

$= h[f_j + \frac{1}{2}\Delta f_j] - \frac{h}{12}\Delta^2 f_j$

$= h[f_j + \frac{1}{2}(f_{j+1} - f_j)] - \frac{h}{12}\Delta^2 f_j$

$\Rightarrow y_{j+1} - y_j = \frac{h}{2}[f_{j+1} + f_j] - \frac{h}{12}\Delta^2 f_j$

$\varepsilon = -\frac{h}{12}\Delta^2 f_j = -\frac{h^3}{12}f''(\xi)$ $x_j \le \xi \le x_{j+1}$